The best we can do against the worst case adversary is
\[
\text{Regret} = \sum_{t=1}^{T} (\hat{y}_t - y_t) = \sum_{t=1}^{T} (\theta_t^\top x_t - y_t) \geq 0,
\]
where $\hat{y}_t$ is the output of the best static predictor $s_t = \arg\min_{s} \sum_{i=1}^{t} x_i^\top s_i$ and $ \hat{\theta}_t$ is the updated parameter estimate.

**Fixed Design Case (previous work)**

**Theorem 1** Assume that $x^T \in \mathbb{R}^d$ is fixed and let $B_t^T$ be a sequence of label bounds. If
\[
x_t^T \in B_t^T \Longleftrightarrow \{x_t^T : B_t \geq \sum_{i=1}^{t-1} x_i^T P_i x_i, B_t \geq 2 \leq t \}
\]
and $y_t \in \mathcal{L}(B_t) = \{y_t : |y_t| \leq B_t \}$, then the minimax strategy calculates states $s_t = \arg\min_{s} \sum_{i=1}^{t} x_i^T s_i$, and $s_t = \sum_{i=1}^{t} x_i x_i^\top$ and plays
\[
\hat{y}_{t+1} = x_{t+1}^T P_{t+1} s_t \tag{MMS}
\]
with coefficient matrices defined by
\[
P_T = \Pi_T^\top \text{ and recursion } P_t = P_{t+1} + P_{t+1} x_{t+1} x_{t+1}^T P_{t+1}.
\]

- Can compute $V_t(y_t)$ by backwards induction efficiently
- Applying Sherman-Morrison yields
\[
P_t = \Pi_t + \sum_{i=t+1}^{T} x_i^T P_{i+1} x_i x_i^\top \Pi_i \geq \Pi_t.
\]

**Forward Recursion**

- How can we generalize to adversarial design? Looks hard:
  - For fixed-design, every $P_t$ depends on all $x^T$
  - Full backwards induction with max$_{x_0}$ is intractable

- Key observation: we can invert the recursion. Given base case $\Sigma_0 = P_0^\top$, define
\[
P_1 = P_{t+1} - a_t \frac{1}{x_t^\top P_{t+1} x_t} P_{t+1} x_t x_t^\top P_{t+1} \tag{1}
\]
for $b_t^2 = x_t^\top P_{t+1} x_t$, $a_t := \left(\sqrt{b_t^2} + 1 - 1\right) \left(\sqrt{b_t^2} + 1 + 1\right)^{-1}$

- Treat $\Sigma$ as a covariate budget for the adversary:
\[
A(\Sigma) := \{x^T : \text{for } P_0, \ldots, P_T \text{ defined by (1), } P_0 \leq \Sigma \}
\]
\[
= \{x^T : P_0 = \Sigma \text{ and } P_t \geq \Pi_t, \forall t \geq 1\}
\]

- If the adversary plays $x^T$ with $P_0(x^T) = \Sigma$, then we responded optimally!

**Hold Your Horses**

- In general, constraining $x^T \in A(\Sigma)$ is not sufficient
- For a finite sequence, the regret of $x^T$ can be $O(\log(T))$

---

**In 1-D**

- Round 1
- Round 2
- Round 3
- Loss of the best

**Minimax Regret**

The best we can do against the worst case adversary is
\[
\left(\max_{\hat{\theta}_t} \max_{\theta_t} \sum_{t=1}^{T} (\hat{y}_t - y_t)^2 - \min_{\theta_t} \sum_{t=1}^{T} (\theta_t^\top x_t - y_t)^2\right) \text{calculated by the backward induction (for known } T)
\]

\[
V_T(\hat{x}_t^T y_t^T) = -\min_{\hat{\theta}_t} \sum_{t=1}^{T} (\hat{\theta}_t^\top x_t - y_t)^2
\]

\[
V_{t-1}(\hat{x}_{t-1}^T y_{t-1}^T) = \max_{\hat{\theta}_t} \max_{\theta_t} \sum_{t=1}^{T} (\hat{\theta}_t^\top x_t - y_t)^2 + V_t(\hat{x}_t^T y_t^T)
\]

**Proof Intuition**

- The proof defines an “early stopping game”, where $x_t^T$ are fixed but the adversary can stop any round
- We calculate the difference in regret between stopping at $t$ and continuing until $T$
- Shorter games may cause more regret because (MMS) is over-regularizing
- The C condition holds when the adversary always causes more regret by continuing
- Under C, the adversary wants to play out the budget, which is the setting where (MMS) is optimal

**Fixed Design Case (previous work)**

- Define label budget $\gamma_0 > 0$ with update $\gamma_t = \gamma_{t-1} - B_t^2 x_t^T P_t x_t$
- Define the continuation conditions
\[
\mathcal{C}(\Sigma, \gamma_t) := \{x_t^T : s_t^2 (\Pi_t - P_t) s_t \leq \gamma_t \forall t \geq 0, \forall s_t\},
\]

where $s_t$ ranges over all $x_t^T \in A(\Sigma) \cap B(B_t, \Sigma)$ and $y_t^T \in \mathcal{L}(B_t)$.

**Theorem 2** For any $\{B_t\} > 0$, $\Sigma > 0$ and $\gamma_0 > 0$, (MMS) with $P_t$ defined by (1) has minimax regret $\gamma_0$ and is horizon-independent minimax optimal; that is,
\[
\sup_{\gamma_t} \left\{\inf_{P_t} \sup_{\gamma_t} \mathcal{C}(\Sigma, \gamma_t) : \mathcal{C}(\Sigma, \gamma_t) \cap \mathcal{C}(\Sigma, \gamma_t) \cap \mathcal{C}(\Sigma, \gamma_t) \right\} = 0,
\]
over $x_t^T \in A(\Sigma) \cap B(B_t, \Sigma)$ and $y_t^T \in \mathcal{L}(B_t)$.

- Removing $A$, $B$, or $C$ lets the adversary cause infinite regret
- We can compete with strategies that know $T$ and $\gamma_0$
- The game-length measure $T$ is replaced with the more natural measure $\Sigma$