Minimax Strategies for Online Linear Regression, Square Loss Prediction, and Time-Series Prediction

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Square loss protocol

Convex set $C$, length $T$, and know loss functions $\ell$. For each round $t = 1, \ldots, T$,

- We play $a_t \in C$
- Nature reveals $y_t \in C$
- We incur loss

$$\ell(a_t, y_t) = \|a_t - y_t\|^2$$

For some comparator class $\mathcal{A}$, the best comparator is

$$L_T^*(y_1^T) = \min_{a \in \mathcal{A}} \sum_{t=1}^{T} \ell(a, y_t).$$

Goal: find a strategy with minimum regret

$$\text{Regret} := \sum_{t=1}^{T} \ell(a_t, y_t) - L_T^*(y_1^T)$$
Section 1

What is Minimax?
What is minimax?

In Game Theory,

\[ V = \inf_{A \in \text{Strategies}} \sup_{Y \in \text{Strategies}} \text{Regret}(A, Y) \]

What are strategies?

\[ A = \{ g_t : (a_{t-1}, s_{t-1}) \rightarrow a_t \} \]
\[ Y = \{ f_t : (a_{t-1}, s_{t-1}) \rightarrow y_t \} \]

How can we solve for \( g_t \) and \( f_t \)?
What is minimax?

In Game Theory

\[ V := \inf_{A \in \text{Strategies}} \sup_{Y \in \text{Strategies}} \text{Regret}(A, Y) \]
What is minimax?

- In Game Theory

\[ V := \inf_{A \in \text{Strategies}} \sup_{Y \in \text{Strategies}} \text{Regret}(A, Y) \]

- What are strategies?

\[ A = \{ g_t : (a_{t-1}^1, s_{t-1}^1) \rightarrow a_t \} \]
\[ Y = \{ f_t : (a_t^1, s_{t-1}^1) \rightarrow y_t \} \]
What is minimax?

▶ In Game Theory

\[ V := \inf_{\mathcal{A} \in \text{Strategies}} \sup_{\mathcal{Y} \in \text{Strategies}} \text{Regret}(\mathcal{A}, \mathcal{Y}) \]

▶ What are strategies?

\[ \mathcal{A} = \{ g_t : (a_1^{t-1}, s_1^{t-1}) \rightarrow a_t \} \]
\[ \mathcal{Y} = \{ f_t : (a_1^t, s_1^{t-1}) \rightarrow y_t \} \]

▶ How can we solve for \( g_t \) and \( f_t \)?
Value of the game

\[ \sum_{t=1}^{T} \ell(a_t, y_t) - L^*(y_1^T) \]
Value of the game

\[
\min_{a_T} \sum_{t=1}^{T} \ell(a_t, y_t) - L^*(y_1^T)
\]
Value of the game

$$\min_{a_T} \max_{y_T} \sum_{t=1}^{T} \ell(a_t, y_t) - L^*(y_T^1)$$
Value of the game

\[
\min_{a_{T-1}} \max_{y_{T-1}} \min_{a_T} \max_{y_T} \sum_{t=1}^{T} \ell(a_t, y_t) - L^*(y^T_1)
\]
Value of the game

\[ V := \min_{a_1} \max_{y_1} \cdots \min_{a_{T-1}} \max_{y_{T-1}} \min_{a_T} \max_{y_T} \sum_{t=1}^{T} \ell(a_t, y_t) - L^*(y_T^1) \]
Value of the game

\[ V := \min_{a_1} \max_{y_1} \cdots \min_{a_{T-1}} \max_{y_{T-1}} \min_{a_T} \max_{y_T} \sum_{t=1}^{T} \ell(a_t, y_t) - L^*(y_T^1) \]

- \( a_t \) can be a function of \( a_1^{t-1} \) and \( y_1^{t-1} \) only
- \( y_t \) can be a function of \( a_1^t \) and \( y_1^{t-1} \) only
Value of the game

\[ V := \min_{a_1} \max_{y_1} \cdots \min_{a_{T-1}} \max_{y_{T-1}} \min_{a_T} \max_{y_T} \sum_{t=1}^{T} \ell(a_t, y_t) - L^*(y_T^1) \]

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- \( y_t \) can be a function of \( a_1^t \) and \( y_1^{t-1} \) only
- Value is the regret when both players are optimal
Value of the game

\[ V := \min_{a_1} \max_{y_1} \cdots \min_{a_{T-1}} \max_{y_{T-1}} \min_{a_T} \max_{y_T} \sum_{t=1}^{T} \ell(a_t, y_t) - L^*(y_1^T) \]

- \( a_t \) can be a function of \( a_1^{t-1} \) and \( y_1^{t-1} \) only
- \( y_t \) can be a function of \( a_1^t \) and \( y_1^{t-1} \) only
- Value is the regret when both players are optimal
- How can we compute them?
Backwards induction

\[ \sum_{t=1}^{T} \ell(a_t, y_t) - L^*(y_T) \]
Backwards induction

\[
\min_{a_T} \max_{y_T} \sum_{t=1}^{T} \ell(a_t, y_t) - L^*(y_1^T)
\]
Backwards induction

\[
\min_{a_T} \max_{y_T} \sum_{t=1}^{T} \ell(a_t, y_t) - L^*(y_T^1) \\
= \sum_{t=1}^{T-1} \ell(a_t, y_t) + \min_{a_T} \max_{y_T} \ell(a_T, y_T) - L^*(y_T^1)
\]
Backwards induction

\[
\min_{a_T} \max_{y_T} \sum_{t=1}^{T} \ell(a_t, y_t) - L^*(y_1^T)
\]

\[
= \sum_{t=1}^{T-1} \ell(a_t, y_t) + \min_{a_T} \max_{y_T} \ell(a_T, y_T) - L^*(y_1^T)
\]

\[
= \sum_{t=1}^{T-1} \ell(a_t, y_t) + V_{T-1}(y_1^{T-1})
\]
Backwards induction

\[\min_{a_{T-1}} \max_{y_{T-1}} \min_{a_T} \max_{y_T} \sum_{t=1}^{T} \ell(a_t, y_t) - L^*(y_T^T)\]

\[= \min_{a_{T-1}} \max_{y_{T-1}} \sum_{t=1}^{T-1} \ell(a_t, y_t) + V_{T-1}(y_{T-1}^{T-1})\]
Backwards induction

\[
\min \max_{a_{T-1}} \min \max_{y_{T-1}} \sum_{t=1}^{T} \ell(a_t, y_t) - L^*(y_1^T)
\]

\[
= \min \max_{a_{T-1}} \sum_{t=1}^{T-1} \ell(a_t, y_t) + V_{T-1}(y_1^{T-1})
\]

\[
= \sum_{t=1}^{T-2} \ell(a_t, y_t) + \min \max_{a_{T-1}} \ell(a_{T-1}, y_{T-1}) + V_{T-1}(y_1^{T-1})
\]
Backwards induction

\[
\min_{a_{T-1}} \max_{y_{T-1}} \min_{a_T} \max_{y_T} \sum_{t=1}^{T} \ell(a_t, y_t) - L^*(y_1^T)
\]

\[
= \min_{a_{T-1}} \max_{y_{T-1}} \sum_{t=1}^{T-1} \ell(a_t, y_t) + V_{T-1}(y_1^{T-1})
\]

\[
= \sum_{t=1}^{T-2} \ell(a_t, y_t) + \min_{a_{T-1}} \max_{y_{T-1}} \sum_{t=1}^{T-1} \ell(a_t, y_t) + V_{T-1}(y_1^{T-1})
\]

\[
= \sum_{t=1}^{T-2} \ell(a_t, y_t) + V_{T-2}(y_1^{T-2})
\]
Backwards induction

\[
\min_{a_1} \max_{y_1} \cdots \min_{a_{T-1}} \max_{y_{T-1}} \min_{a_T} \max_{y_T} \sum_{t=1}^{T} \ell(a_t, y_t) - L^*(y_1^T)
\]

= \min_{a_1} \max_{y_1} \cdots \sum_{t=1}^{T-2} \ell(a_t, y_t) + V_{T-2}(y_1^{T-2})
Backwards induction

$$\min_{a_1} \max_{y_1} \cdots \min_{a_{T-1}} \max_{y_{T-1}} \min_{a_T} \max_{y_T} \sum_{t=1}^{T} \ell(a_t, y_t) - L^*(y_1^{T})$$

$$= \min_{a_1} \max_{y_1} \cdots \sum_{t=1}^{T-2} \ell(a_t, y_t) + V_{T-2}(y_1^{T-2})$$

$$\cdots$$
Value-to-go

Inductive definition:

\[ V_T(y_1^T) := -L_T^*(y_1^T) \quad (1) \]
\[ V_{t-1}(y_1^{t-1}) := \min_{a_t} \max_{y_t} \ell(a_t, y_t) + V_t(y_1^{t-1}, y_t) \quad (2) \]
Value-to-go

Inductive definition:

\[
V_T(y_1^T) := -L^*_T(y_1^T) \tag{1}
\]

\[
V_{t-1}(y_1^{t-1}) := \min_{a_t} \max_{y_t} \ell(a_t, y_t) + V_t(y_1^{t-1}, y_t) \tag{2}
\]

The minimax regret \( V \) equals value-to-go \( V_0(\epsilon) \) (empty history).

The minimax strategy: after seeing \( y_1, \ldots, y_{t-1} \),

- Compute \( V_t(y_1, \ldots, y_t) \)
- Choose \( a_t \) as the minimizer of

\[
\max_{y_t} \ell(a_t, y_t) + V_t(y_1^{t-1}, y_t)
\]
Value-to-go

Inductive definition:

\[ V_T(y_1) := -L_T^*(y_1) \]  
\[ V_{t-1}(y_1^{t-1}) := \min_{a_t} \max_{y_t} \ell(a_t, y_t) + V_t(y_1^{t-1}, y_t) \]  \hspace{1cm} (2)

The minimax regret \( V \) equals value-to-go \( V_0(\epsilon) \) (empty history).

The minimax strategy: after seeing \( y_1, \ldots, y_{t-1} \),

- Compute \( V_t(y_1, \ldots, y_t) \)
- Choose \( a_t \) as the minimizer of

\[ \max_{y_t} \ell(a_t, y_t) + V_t(y_1^{t-1}, y_t) \]

Problem: this is expensive (usually exponentially so).
Outline

- What is minimax?
- Two square loss games
- Fixed-design online linear regression
- Time series prediction
Section 2

Square loss game
Square loss protocol (with Koolen and Bartlett)

Convex set $C$, length $T$, and know loss functions $\ell$.

For each round $t = 1, \ldots, T$,

- We play $a_t \in C$
- Nature reveals $y_t \in C$
- We incur loss $\ell(a_t, y_t) := \|a_t - y_t\|^2_W = (a_t - y_t)^\top W^{-1} (a_t - y_t)$

Our goal is to minimize regret w.r.t. best fixed action $a$ in hindsight

$$\text{Regret} := \sum_{t=1}^T \ell(a_t, y_t) - \min_a \sum_{t=1}^T \ell(a, y_t)$$
Solving the minimax strategy

Value-to-go:

\[ V_T(y_1^T) := -L^*_T(y_1^T) \]  \hspace{1cm} (1)

\[ V_{t-1}(y_{1}^{t-1}) := \min_{a_t} \max_{y_t} \ell(a_t, y_t) + V_t(y_{1}^{t-1}, y_t) \]  \hspace{1cm} (2)

▶ Using sufficient statistics

\[ s_t = \sum_{\tau=1}^{t} y_{\tau} \quad \text{and} \quad \sigma^2_t = \sum_{\tau=1}^{t} y_{\tau}^\top W^{-1} y_{\tau} \]

▶ First, we need \( L^*_T(y_1^T) \):

\[ L^*_T = \inf_{a \in \mathbb{R}^d} \sum_{t=1}^{T} \|a - y_t\|^2_W = \sigma^2_T - \frac{1}{T} s_T^\top W^{-1} s_t \]

and the minimizer is the mean outcome \( a^* = \frac{1}{T} \sum_{t=1}^{T} y_t \).
Calculating the value function for $C = \triangle$

Value-to-go:

\[
V_T(y_1^T) := -L^*_T(y_1^T) \quad (1)
\]

\[
V_{t-1}(y_{1}^{t-1}) := \min_{a_t} \max_{y_t} \ell(a_t, y_t) + V_t(y_{1}^{t-1}, y_t) \quad (2)
\]

- Base case: $V_T(y_1^T) = -L^*_T = \frac{1}{T}s_T W^{-1}s_T - \sigma^2_T$
Calculating the value function for $C = \triangle$

Value-to-go:

$$V_T(y_1^T) := -L^*_T(y_1^T)$$ (1)

$$V_{t-1}(y_{1}^{t-1}) := \min_{a_t} \max_{y_t} \ell(a_t, y_t) + V_t(y_{1}^{t-1}, y_t)$$ (2)

- Base case: $V_T(y_1^T) = -L^*_T = \frac{1}{T} s_T W^{-1} s_T - \sigma^2_T$
- "Guess":

$$V_t(s_t, \sigma^2_t) = \alpha_t s_t^T W^{-1} s_t - \sigma^2_t + b^T s_t + \gamma_t,$$
Calculating the value function for $C = \triangle$

Value-to-go:

\[ V_T(y_1^T) := -L^*_T(y_1^T) \] (1)
\[ V_{t-1}(y_1^{t-1}) := \min_{a_t} \max_{y_t} \ell(a_t, y_t) + V_t(y_1^{t-1}, y_t) \] (2)

- **Base case:** $V_T(y_1^T) = -L^*_T = \frac{1}{T} s_T^T W^{-1} s_T - \sigma^2_T$
- **“Guess”:**

\[ V_t(s_t, \sigma^2_t) = \alpha_t s_t^T W^{-1} s_t - \sigma^2_t + b^T s_t + \gamma_t, \]

- **Base case:** $\alpha_T = \frac{1}{T}, \gamma_t = 0$
Calculating the value function for $\mathcal{C} = \triangle$

Value-to-go:

$$V_T(y_1^T) := -L_T^*(y_1^T) \quad (1)$$

$$V_{t-1}(y_1^{t-1}) := \min_{a_t} \max_{y_t} \ell(a_t, y_t) + V_t(y_1^{t-1}, y_t) \quad (2)$$

- **Base case:** $V_T(y_1^T) = -L_T^* = \frac{1}{T} s_T^T W^{-1} s_T - \sigma^2_T$
- **“Guess”:**

$$V_t(s_t, \sigma^2_t) = \alpha t s_t^T W^{-1} s_t - \sigma^2_t + b^T s_t + \gamma_t,$$

- **Base case:** $\alpha_T = \frac{1}{T}$, $\gamma_t = 0$
- **Induction:**

$$V_t(s_t, \sigma^2_t) = \min_{a \in \triangle} \max_{y \in \triangle} \ell(a, y) + V_{t+1}(s_t + y, \sigma^2_t + y^T W^{-1} y)$$
\[ V_t(s_t, \sigma^2_t) = \min_{a \in \Delta} \max_{y \in \Delta} \| a - y \|^2_W + \alpha_t (s_t + y)^T W^{-1} (s_t + y) \\
- (\sigma^2_t + y^T W^{-1} y) + \gamma_t + b^T (s_t + y) \]
\[ V_t(s_t, \sigma^2_t) = \min_{a \in \Delta} \max_{y \in \Delta} \|a - y\|_W^2 + \alpha_t (s_t + y)^\top W^{-1} (s_t + y) \]
\[ - (\sigma^2_t + y^\top W^{-1} y) + \gamma_t + b^\top (s_t + y) \]
\[ = \min_{a \in \Delta} \max_{y \in \Delta} \|a - y\|_W^2 + (\alpha_t - 1) y^\top W^{-1} y + b'^\top y (+c) \]
\[ V_t(s_t, \sigma^2_t) = \min_{a \in \triangle} \max_{y \in \triangle} \|a - y\|^2_W + \alpha_t(s_t + y)^\top W^{-1}(s_t + y) \]
\[- (\sigma^2_t + y^\top W^{-1}y) + \gamma_t + b^\top(s_t + y) \]
\[= \min_{a \in \triangle} \max_{y \in \triangle} \|a - y\|^2_W + (\alpha_t - 1)y^\top W^{-1}y + b'^\top y (+ c) \]
\[= \min_{a \in \triangle} \max_k \|a - e_k\|^2_W + (\alpha_t - 1)e_k^\top W^{-1}e_k + b'^\top e_k \]
\[ V_t(s_t, \sigma^2_t) = \min_{a \in \Delta} \max_{y \in \Delta} \|a - y\|_W^2 + \alpha_t(s_t + y)^\top W^{-1} (s_t + y) \]
\[ - (\sigma^2_t + y^\top W^{-1} y) + \gamma_t + b^\top (s_t + y) \]
\[ = \min_{a \in \Delta} \max_{y \in \Delta} \|a - y\|_W^2 + (\alpha_t - 1)y^\top W^{-1} y + b^\top y (+c) \]
\[ = \min_{a \in \Delta} \max_k \|a - e_k\|_W^2 + (\alpha_t - 1)e_k^\top W^{-1} e_k + b^\top e_k \]
\[ = \max_p \min_{a \in \Delta} E_{k \sim p} \left[ \|a - e_k\|_W^2 + (\alpha_t - 1)e_k^\top W^{-1} e_k + b^\top e_k \right] \]

Easy to solve via Lagrange multipliers.
$$V_t(s_t, \sigma^2_t) = \min_{a \in \triangle} \max_{y \in \triangle} \|a - y\|^2_W + \alpha_t(s_t + y)^\top W^{-1}(s_t + y)$$

$$- (\sigma^2_t + y^\top W^{-1}y) + \gamma_t + b^\top (s_t + y)$$

$$= \min_{a \in \Delta} \max_{y \in \Delta} \|a - y\|^2_W + (\alpha_t - 1)y^\top W^{-1}y + b'^\top y( + c)$$

$$= \min_{a \in \Delta} \max_k \|a - e_k\|^2_W + (\alpha_t - 1)e_k^\top W^{-1}e_k + b'^\top e_k$$

$$= \max_p \min_{a \in \Delta} \mathbb{E}_{k \sim p} \left[ \|a - e_k\|^2_W + (\alpha_t - 1)e_k^\top W^{-1}e_k + b'^\top e_k \right]$$

$$= \max_p -p^\top W^{-1} p + (\alpha_t \text{ diag}(W^{-1}) + b')^\top p$$

Easy to solve via Lagrange multipliers.
\[ V_t(s_t, \sigma^2_t) = \min_{a \in \Delta} \max_{y \in \Delta} \|a - y\|_W^2 + \alpha_t(s_t + y)^T W^{-1}(s_t + y) \]

\[- (\sigma^2_t + y^T W^{-1} y) + \gamma_t + b^T(s_t + y) \]

\[= \min_{a \in \Delta} \max_{y \in \Delta} \|a - y\|_W^2 + (\alpha_t - 1)y^T W^{-1} y + b^T y (+c) \]

\[= \min_{a \in \Delta} \max_k \|a - e_k\|_W^2 + (\alpha_t - 1)e_k^T W^{-1} e_k + b^T e_k \]

\[= \max_p \min_{a \in \Delta} \mathbb{E}_{k \sim p} \left[ \|a - e_k\|_W^2 + (\alpha_t - 1)e_k^T W^{-1} e_k + b^T e_k \right] \]

\[= \max_p -p^T W^{-1} p + (\alpha_t \text{ diag}(W^{-1}) + b')^T p \]

Easy to solve via Lagrange multipliers.
Simplex game (e.g. Brier game)

Theorem

Let $C = \triangle$. For $W$ satisfying an alignment condition, the value-to-go is

$$V_t(s_t, \sigma^2_t) = \alpha_t s_t^T W^{-1} s_t - \sigma^2_t + (1 - t\alpha_t) \text{diag}(W^{-1})^T (s_t + \text{const})$$

with coefficients

$$\alpha_T = \frac{1}{T} \text{ and } \alpha_t = \alpha_{t+1}^2 + \alpha_{t+1}.$$ 

The minimax and maximin strategies are

$$a_t = p_t = \frac{s_t}{t} t\alpha_{t+1} + c(1 - t\alpha_{t+1})$$

which is data mean $\frac{s_t}{t}$ shrunk towards center

$$c = \frac{W^1}{1^T W^1} + (W - \frac{W^{11^T} W}{1^T W^1}) \text{diag}(W^{-1}).$$
Theorem

Let $C = \bigcirc$. For any positive definite $W$ the value-to-go is

$$V_t(s_t, \sigma^2_t) = s_t^\top A_t s - \sigma^2_t + \text{const}.$$  

For round $t$, the minimax strategy plays

$$a^* = \left( \lambda_{\max}(A_t) I - (A_t - W^{-1}) \right)^{-1} A_t s$$

with coefficients $A_T = \frac{1}{t} W^{-1}$ and

$$A_{t-1} = A_t \left( W^{-1} + \lambda_{\max}(A_t) I - A_t \right)^{-1} A_t + A_t.$$
Regret bounds

\begin{itemize}
\item \(\text{Regret}_{\text{Brier}} \propto \sum_{t=1}^{T} \alpha_t.\)
\item \(\text{Regret}_{\text{Ball}} = \lambda_{\text{max}}(W^{-1}) \sum_{t=1}^{T} \alpha_t.\)
\item [1] show that \(\sum_{t=1}^{T} \alpha_t = \log(T) - \log \log(T) + O\left(\frac{\log(T)}{\log \log(T)}\right).\)
\item Compare with \(O(\log(T))\) of Follow the Leader.
\end{itemize}

E. Takimoto, M. Warmuth
The minimax strategy for Gaussian density estimation
In \textit{COLT '00}
Section 3

Online Linear regression
Online linear regression (with Bartlett, Koolen, Takimoto, Warmuth)

Fix a covariate sequence $\mathbf{x}_1, \ldots, \mathbf{x}_T$ (fixed design) and length $T$.
For each round $t = 1, \ldots, T$,

- We play $a_t \in \mathbb{R}$
- Nature reveals $y_t \in [-B_t, B_t]$
- We incur loss
  \[ \ell(a_t, y_t) = (a_t - y_t)^2 \]
- Minimax Regret is
  \[
  \min_{a_1} \max_{y_1} \cdots \min_{a_T} \max_{y_T} \sum_{t=1}^{T} (a_t - y_t)^2 - \min_{\theta \in \mathbb{R}^d} \sum_{t=1}^{T} (\theta^T \mathbf{x}_t - y_t)^2
  \]
Online linear regression (with Bartlett, Koolen, Takimoto, Warmuth)

Fix a covariate sequence $x_1, \ldots, x_T$ (fixed design) and length $T$. For each round $t = 1, \ldots, T$,

- We play $a_t \in \mathbb{R}$
- Nature reveals $y_t \in [-B_t, B_t]$
- We incur loss
  $$\ell(a_t, y_t) = (a_t - y_t)^2$$
- Minimax Regret is
  $$\min_{a_1} \max_{y_1} \cdots \min_{a_T} \max_{y_T} \sum_{t=1}^{T} (a_t - y_t)^2 - \min_{\theta \in \mathbb{R}^d} \sum_{t=1}^{T} (\theta^T x_t - y_t)^2$$

algorithm
Online linear regression (with Bartlett, Koolen, Takimoto, Warmuth)

Fix a covariate sequence \( x_1, \ldots, x_T \) (fixed design) and length \( T \).
For each round \( t = 1, \ldots, T \),

- We play \( a_t \in \mathbb{R} \)
- Nature reveals \( y_t \in [-B_t, B_t] \)
- We incur loss
  \[
  \ell(a_t, y_t) = (a_t - y_t)^2
  \]
- Minimax Regret is

\[
\min_{a_1} \max_{y_1} \ldots \min_{a_T} \max_{y_T} \left\{ \sum_{t=1}^{T} (a_t - y_t)^2 \right\} - \min_{\theta \in \mathbb{R}^d} \sum_{t=1}^{T} (\theta^T x_t - y_t)^2
\]
Solving the value-to-go

Value-to-go:

\[ V_T(y^T_1) := -L^*_T(y^T_1) \]  
\[ V_{t-1}(y^{t-1}_1) := \min_{a_t} \max_{y_t} \ell(a_t, y_t) + V_t(y^{t-1}_1, y_t) \]

Define

\[ s_t = \sum_{\tau=1}^{t} y_{\tau} x_{\tau}, \quad \sigma^2_t = \sum_{\tau=1}^{t} y^2_{\tau}, \quad P_T = \left( \sum_{t=1}^{T} x_t x_t^\top \right)^{-1} \]

Base case is ordinary least squares: \( \theta^* = P_T s_T \) and

\[ V_T(y^T_1) = -L^*_T(y^T_1) = \min_{\theta \in \mathbb{R}^d} \sum_{t=1}^{T} (\theta^\top x_t - y_t)^2 \]

\[ = s_T^\top P_T s_T - \sigma^2_T. \]
Induction Hypothesis

\[ V_t(s_t, \sigma_t^2) = s_t^T P_t s_t - \sigma_t^2 + \gamma_t, \]
Induction Hypothesis

\[ V_t(s_t, \sigma_t^2) = s_t^T P_t s_t - \sigma_t^2 + \gamma_t, \]

Backwards induction:

\[
V_t(s_t, \sigma_t^2) := \min_{a_{t+1}} \max_{y_{t+1}} \left( a_{t+1} - y_{t+1} \right)^2 \\
+ (s_t + y_{t+1} x_{t+1})^T P_{t+1} (s_t + y_{t+1} x_{t+1}) \\
- (\sigma_t^2 + y_{t+1}^2) + \gamma_{t+1} \\
= \min_{a_{t+1}} \max_{y_{t+1}} a_{t+1}^2 + 2y_{t+1} \left( x_{t+1}^T P_{t+1} s_t - a_{t+1} \right) \\
+ \left( x_{t+1}^T P_{t+1} x_{t+1} \right) y_{t+1}^2 + \text{const}
\]
Induction Hypothesis

\[ V_t(s_t, \sigma^2_t) = s_t^T P_t s_t - \sigma^2_t + \gamma_t, \]

Backwards induction:

\[ V_t(s_t, \sigma^2_t) := \min_{a_{t+1}} \max_{y_{t+1}} \left( a_{t+1} - y_{t+1} \right)^2 \]

\[ + (s_t + y_{t+1} x_{t+1})^T P_{t+1} (s_t + y_{t+1} x_{t+1}) \]

\[ - (\sigma^2_t + y_{t+1}^2) + \gamma_{t+1} \]

\[ = \min_{a_{t+1}} \max_{y_{t+1}} a_{t+1}^2 + 2y_{t+1} (x_{t+1}^T P_{t+1} s_t - a_{t+1}) \]

\[ + (x_{t+1}^T P_{t+1} x_{t+1}) y_{t+1}^2 + \text{const} \]
**Induction Hypothesis**

\[ V_t(s_t, \sigma^2_t) = s_t^T P_t s_t - \sigma^2_t + \gamma_t, \]

**Backwards induction:**

\[
V_t (s_t, \sigma^2_t) := \min_{a_{t+1}} \max_{y_{t+1}} \left( a_{t+1} - y_{t+1} \right)^2 \\
+ (s_t + y_{t+1} x_{t+1})^T P_{t+1} (s_t + y_{t+1} x_{t+1}) \\
- (\sigma^2_t + y^2_{t+1}) + \gamma_{t+1}
\]

\[
= \min_{a_{t+1}} \max_{y_{t+1}} a_{t+1}^2 + 2y_{t+1} (x_{t+1}^T P_{t+1} s_t - a_{t+1}) \\
+ (x_{t+1}^T P_{t+1} x_{t+1}) y_{t+1}^2 + \text{const}
\]

**This is convex in** \( y_{t+1} \) **and hence** \( y_{t+1} = \pm B_{t+1} \), so

\[
V_t (s_t, \sigma^2_t) = \min_{a_{t+1}} a_{t+1}^2 + 2B_{t+1} \left| x_{t+1}^T P_{t+1} s_t - a_{t+1} \right| \\
+ (x_{t+1}^T P_{t+1} x_{t+1}) B^2 + \text{const}
\]
We had

\[ V_t (s_t, \sigma_t^2) = \min_{\alpha_{t+1}} \alpha_{t+1}^2 + 2B_{t+1} \left| x_{t+1}^T P_{t+1} s_t - \alpha_{t+1} \right| \]

\[ + \left( x_{t+1}^T P_{t+1} x_{t+1} \right) B^2 + \text{const} \]
We had

\[ V_t(s_t, \sigma_t^2) = \min_{\alpha_{t+1}} \frac{1}{2} \alpha_{t+1}^2 + 2B_{t+1} \left| x_{t+1}^T P_{t+1} s_t - \alpha_{t+1} \right| \]

\[ + \left( x_{t+1}^T P_{t+1} x_{t+1} \right) B^2 + \text{const} \]

If \( |x_{t+1}^T P_{t+1} s_t| \leq B_{t+1} \), setting subgradient to 0 yields

\[ \alpha_{t+1} = x_{t+1}^T P_{t+1} s_t \]
We had

\[
V_t \left( s_t, \sigma_t^2 \right) = \min_{a_{t+1}} a_{t+1}^2 + 2B_{t+1} \left| x_{t+1}^T P_{t+1} s_t - a_{t+1} \right|
\]

\[
+ \left( x_{t+1}^T P_{t+1} x_{t+1} \right) B^2 + \text{const}
\]

If \(|x_{t+1}^T P_{t+1} s_t| \leq B_{t+1}\), setting subgradient to 0 yields

\[
a_{t+1} = x_{t+1}^T P_{t+1} s_t
\]

Plugging in this \(a_{t+1}\), we get

\[
V_t \left( s_t, \sigma_t^2 \right) = s_t^T \left( P_{t+1} x_{t+1} x_{t+1}^T + P_{t+1} + P_{t+1} \right) s_t
\]

\[
- \sigma_t^2 + \gamma_{t+1} + B_{t+1}^2 x_{t+1}^T P_{t+1} x_{t+1}
\]

\[\begin{aligned}
&:= P_t \\
&:= \gamma_t
\end{aligned}\]
We had
\[ V_t(s_t, \sigma_t^2) = \min_{\alpha_{t+1}} \alpha_{t+1}^2 + 2B_{t+1} \left| x_{t+1}^T P_{t+1} s_t - \alpha_{t+1} \right| \]
\[ + \left( x_{t+1}^T P_{t+1} x_{t+1} \right) B^2 + \text{const} \]

If \( |x_{t+1}^T P_{t+1} s_t| \leq B_{t+1} \), setting subgradient to 0 yields
\[ \alpha_{t+1} = x_{t+1}^T P_{t+1} s_t \]

Plugging in this \( \alpha_{t+1} \), we get
\[ V_t(s_t, \sigma_t^2) = s_t^T \left( P_{t+1} x_{t+1} x_{t+1}^T P_{t+1} + P_{t+1} \right) s_t \]
\[ - \sigma_t^2 + \gamma_{t+1} + B_{t+1}^2 x_{t+1}^T P_{t+1} x_{t+1} \]
\[ := \gamma_t \]

Value is
\[ V_0(0, 0) = \gamma_0 = \sum_{t=1}^{T} B_t^2 x_t^T P_t x_t \]
In Summary

Theorem

The strategy

\[ a_{t+1} = x_{t+1}^T P_{t+1} s_t, \]

is minimax optimal and the value-to-go is

\[ V_t(s_t, \sigma^2_t) = s_t^T P_t s_t - \sigma^2_t + \gamma_t, \]

with coefficients

\[ P_T = \left( \sum_{t=1}^{T} x_t x_t^T \right)^{-1}, \quad P_t = P_{t+1} + P_{t+1} x_{t+1} x_{t+1}^T P_{t+1}, \]

\[ \gamma_T = 0, \quad \gamma_t = \gamma_{t+1} + B_{t+1}^2 x_{t+1}^T P_{t+1} x_{t+1}, \]

provided the box constraints \( |x_{t+1}^T P_{t+1} s_t| \leq B_{t+1} \) hold.
Interpretation of $P_t$

$$P_t^{-1} = \sum_{\tau=1}^{t} x_\tau x_\tau^\top + \sum_{\tau=t+1}^{T} \frac{x_\tau^\top P_\tau x_\tau}{1 + x_\tau^\top P_\tau x_\tau} x_\tau x_\tau^\top .$$

- least squares
- re-weighted future instances
Interpretation of $P_t$

$$P_t^{-1} = \sum_{\tau=1}^{t} x_\tau x_\tau^\top + \sum_{\tau=t+1}^{T} \frac{x_\tau^\top P_\tau x_\tau}{1 + x_\tau^\top P_\tau x_\tau} x_\tau x_\tau^\top.$$  

- **least squares**
- **re-weighted future instances**

Recall regret at round $t$: $B_t x_t^\top P_t x_t$
Section 4

Tracking
Time series prediction protocol (with Koolen, Bartlett, Abbasi-Yadkori)

Fix a convex set $C$, length $T$, regularization parameter $\lambda_T$.
For each round $t = 1, \ldots, T$,

- We play $a_t \in C$
- Nature reveals $y_t \in C$
- We incur loss $\ell(a_t, y_t) := \|a_t - y_t\|^2$
- Regret:

$$
\sum_{t=1}^{T} \|a_t - y_t\|^2 - \min_{\hat{a}_1, \ldots, \hat{a}_T} \left\{ \sum_{t=1}^{T} \|\hat{a}_t - y_t\|^2 + \lambda_T \text{tr}(K \hat{A}^\top \hat{A}) \right\}
$$

where $\hat{A} = [\hat{a}_1 \cdots \hat{a}_T]$

- E.g. $\text{tr}(K \hat{A}^\top \hat{A}) = \sum_{t=1}^{T+1} \|\hat{a}_t - \hat{a}_{t-1}\|^2$
Solving the value-to-go (part 3)

Value-to-go:

\[ V_T(y_1^T) := -L^*_T(y_1^T) \]  \hspace{1cm} (1)

\[ V_{t-1}(y_{1}^{t-1}) := \min_{a_t} \max_{y_t} \ell(a_t, y_t) + V_t(y_{1}^{t-1}, y_t) \]  \hspace{1cm} (2)

Histories are \( Y_t = [y_1 \cdots y_t] \).

Offline Problem: \( \hat{A} = Y_T(I + \lambda_T K)^{-1} \) and value

\[ V_T(Y_T) = -L^* = -\text{tr} \left( Y_T(I - (I + \lambda_T K)^{-1}) Y_T^T \right) \]
Behavior of backwards induction

**Theorem**

If $\|b\| \leq 1$, then the minimax problem

$$V^* := \min_a \max_{y: \|y\| \leq 1} \|a - y\|^2 + (\alpha - 1)\|y\|^2 + 2b^T_y$$

has value and minimizer

$$V^* = \begin{cases} \|b\|^2 \quad & \text{if } \alpha \leq 0, \\ \|b\|^2 + \alpha \quad & \text{if } \alpha \geq 0, \end{cases} \quad \text{and} \quad a = \begin{cases} \frac{b}{1-\alpha} \quad & \text{if } \alpha \leq 0, \\ b \quad & \text{if } \alpha \geq 0. \end{cases}$$

Non-trivial induction:

- Curvature of optimization can switch between rounds
- Yet can pre-compute beforehand
Minimax solution

Input: $T$, $K$, $\lambda_T$

Using:
- single-shot game solution, and
- lots of matrix identities

Output: matrices $R_t = \begin{pmatrix} A_t & b_t \\ b_t^T & c_t \end{pmatrix}$

strategy $a_t = X_{t-1} \begin{cases} \frac{b_t}{1 - c_t} & \text{if } c_t \leq 0, \\ b_t - c_t e_t & \text{if } c_t \geq 0. \end{cases}$
Theorem

Under a (typical) no clipping condition on $Y_T$,

$$V_t(Y_t) = \text{tr} (Y_t (R_t - I) Y_t^T) + \sum_{\tau=t+1}^{T} \max\{c_\tau, 0\}$$

and, in the vanilla case (norm bounded data, increments penalized),

$$V = \Theta \left( \frac{T}{\sqrt{1 + \lambda_T}} \right).$$
Section 5

Conclusion
Minimax algorithms can be computationally efficient with enough structure, e.g.
- Normalized Maximum likelihood that is Bayesian
- Certain square losses
- Exploited the fact that saddle point problems with square loss are nice
- Can we characterize the class of functions that are closed w.r.t. the backwards induction operator?
Section 6

Extra slides
The maximin strategy plays two unit length vectors with

\[
\Pr \left( y = a_\perp \pm \sqrt{1 - a_\perp^T a_\perp} v_{\max} \right) = \frac{1}{2} \pm \frac{a_\parallel^T v_{\max}}{2 \sqrt{1 - a_\perp^T a_\perp}},
\]

where \(\lambda_{\max}\) and \(v_{\max}\) correspond to the largest eigenvalue of \(A_{t+1}\) and \(a_\perp\) and \(a_\parallel\) are the components of \(a^*\) perpendicular and parallel to \(v_{\max}\).
Tracking: second order $K$

- Computation: if $K$ and $v_t$ are banded then $R_t^{-1}$ is sparse.
- Here we imposed data bound $\|Y_t v_t\| \leq 1$. In the paper we show that the minimax strategy guarantees an adaptive bound scaling with $\|Y_t v_t\|$.
- A second order smoothness version of $K$ gives complicated $c_t$.

**Figure:** $v_t = e_t - e_{t-1}$

**Figure:** $v_t = e_t - 2e_{t-1} + e_{t-2}$
Ellipse

Fix a budget \( R \geq 0 \), and consider label sequences

\[
\mathcal{Y}_R := \left\{ y_1, \ldots, y_T \in \mathbb{R} : \sum_{t=1}^{T} y_t^2 x_t^\top P_t x_t = R \right\}
\]

We show that \((\text{MM})\) is minimax for this set.

In fact, the regret of \((\text{MM})\) equals

\[
\mathcal{R}_T = \sum_{t=1}^{T} y_t^2 x_t^\top P_t x_t.
\]

This means that this algorithm has two very special properties. First, it is a *strong equalizer* in the sense that it suffers the same regret on all \(2^T\) sign-flips of the labels. And second, it is *adaptive* to the complexity \( R \) of the labels.